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# The Distribution of the Zeros of Certain Orthogonal Polynomials

A. McD. MERCER

*Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario, Canada N1G 2W1*

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If $\alpha_n \in R$, $\beta_n > 0$, $p_{-1} = 0$, and $p_0 = 1$, let $\{p_n\}_0^\infty$ be a sequence of monic polynomials satisfying

$$p_{n+1}(x) = (x - \alpha_n)\, p_n(x) - \beta_n\, p_{n-1}(x) \qquad (n = 0, 1, 2, ...). \tag{1}$$

According to a theorem of Favard [1], there will exist a distribution function $w$ on $(-\infty, +\infty)$ such that

$$\int_{-\infty}^{+\infty} p_m(x)\, p_n(x)\, dw(x) = 0 \qquad (m \neq n).$$

The zeros of $p_n(x)$ are known to be real and distinct and to lie in the smallest closed interval containing the support of $dw$. We will enumerate them as

$$x_1^n < x_2^n < \cdots < x_n^n.$$

Now suppose that $\alpha_n$ and $\beta_n$ are constant ($\alpha_n = \alpha$, $\beta_n = \beta > 0$, say) and that $f$ is a continuous function. The constant-coefficient difference equation (1) is easily solved and the $x_\nu^n$ found. It is then a simple matter to verify that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i^n) = \int_{-\infty}^{+\infty} f(t)\, d\phi(t),$$

where (2)

$$\phi(t) = \frac{1}{\pi} \int_{-\infty}^{t} \frac{\Phi(\alpha - 2\sqrt{\beta}, \alpha + 2\sqrt{\beta})}{[4\beta - (u - \alpha)^2]^{1/2}}\, du.$$



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Here $\Phi(a, b)$ denotes the characteristic function of the interval $(a, b)$. Note that the support of $d\phi$ is $[\alpha - 2\sqrt{\beta}, \alpha + 2\sqrt{\beta}]$ and that all of the $x_i^n$ lie in this interval, so that the result could be stated equivalently using a function of the type

$$\psi_x(t) = 1 \ (t \leqslant x) := 0 \quad (t > x)$$

in place of the continuous $f$.

If, instead of supposing that $\alpha_n = \alpha$, $\beta_n = \beta > 0$ for all $n$, we relax this condition of constancy to $\alpha_n \to \alpha$, $\beta_n \to \beta > 0$ it turns out that the result (2) continues to hold. This result is not elementary and several proofs have been given [2–5].

In the present note it is our purpose to show that the condition of constancy of the sequences $\{\alpha_n\}_0^\infty$ and $\{\beta_n\}_0^\infty$ can be relaxed in yet another fashion when, once again the result (2) is obtained. We will need the following definition:

DEFINITION. A sequence $\{\alpha_n\}_0^\infty$ will be called *nearly constant* if a constant $\alpha$ exists such that

$$\#\{\alpha_k : 0 \leqslant k \leqslant n : \alpha_k \neq \alpha\} = o(n) \qquad \text{as} \quad n \to \infty.$$

Our object, then, is to prove the following result:

THEOREM. *If in* (1) *the sequences* $\{\alpha_n\}_0^\infty$ *and* $\{\beta_n\}_0^\infty$ *are nearly constant, these constants being* $\alpha$ *and* $\beta > 0$, *respectively, then* (2) *holds provided* $f \in C(-\infty, +\infty)$ *and the limits* $\lim_{x \to \pm\infty} f(x)$ *exist.*

*Proof of the Theorem.* We will denote by $p_n(x)$ the polynomials arising from the nearly constant sequences $\{\alpha_n\}_0^\infty$ and $\{\beta_n\}_0^\infty$ and by $q_n(x)$ the polynomials corresponding to $\alpha_n = \alpha$, $\beta_n = \beta$ for all $n$. Also, we will work with the function $\psi_x(t)$ rather than with a continuous $f(t)$. The general case will follow from this.

We introduce two infinite, tridiagonal, symmetric matrices $A$ and $B$ as follows:

$$A = \begin{bmatrix} \alpha_0 & \gamma_0 & & & \\ \gamma_0 & \alpha_1 & \gamma_1 & & \\ & \gamma_1 & \alpha_2 & \gamma_2 & \\ & & & \ddots & \\ & & & & \end{bmatrix}, \qquad B = \begin{bmatrix} \alpha & \gamma & & & \\ \gamma & \alpha & \gamma & & \\ & \gamma & \alpha & \gamma & \\ & & & \ddots & \\ & & & & \end{bmatrix}$$

in which $\gamma_k = \sqrt{\beta_k}$, $\gamma = \sqrt{\beta}$. The principal $n \times n$ section of $A$, for example, will be denoted by $A_n$. Then it is well known that

$$\det(A_n - xI_n) = (-1)^n p_n(x)$$
$$\det(B_n - xI_n) = (-1)^n q_n(x)$$

so that the zeros of $p_n(x)$ and $q_n(x)$ are, respectively, the eigenvalues of $A_n$ and of $B_n$. We denote these by

$$a_1^n < a_2^n < \cdots < a_n^n$$

and

$$b_1^n < b_2^n < \cdots < b_n^n.$$

We see at once that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \psi_x(b_i^n) = \int_{-\infty}^{+\infty} \psi_x(t)\, d\phi(t) \qquad (= L(\psi_x), \text{ say}). \tag{3}$$

This is the elementary case of (2) mentioned at the outset of this note. Our object is to prove that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \psi_x(a_i^n) = L(\psi_x)$$

also. The sequence $\{\alpha_n\}_0^\infty$ is nearly constant $(=\alpha)$ and so we proceed down the main diagonal of the matrix $A_n$ and delete any row and column which contains a diagonal element which is not equal to $\alpha$. This process is repeated with the other two diagonals of $A_n$, deleting any row and column which contains an element on that diagonal which is not equal to $\gamma$. The $(n-k) \times (n-k)$ matrix which finally results will be called $C_{n-k}$ and clearly $k = o(n)$ as $n \to \infty$ in view of our definition of "nearly constant."

On examining the form of $C_{n-k}$ it is found to consist of blocks of square matrices whose diagonal elements form the diagonal elements of $C_{n-k}$ and each of these square matrices is a $B_q$ $(1 \leqslant q \leqslant n-k)$. The eigenvalues of $C_{n-k}$ will be written as

$$c_1^{n-k} \leqslant c_2^{n-k} \leqslant \cdots \leqslant c_{n-k}^{n-k}.$$

Since $C_{n-k}$ was obtained by deleting $k$ rows and corresponding columns of $A_n$ it is classical [6, 7] that their eigenvalues are related as follows:

$$c_i^{n-k} \in [a_i^n, a_{i+k}^n] \qquad (i = 1, 2, ..., n-k).$$

From this it follows that

$$\text{(The number of eigenvalues of } A_n \leqslant x) \\ - \text{(The number of eigenvalues of } C_{n-k} \leqslant x) = J_k(x),$$

where

$$0 \leqslant J_k(x) \leqslant k. \tag{4}$$

This can be written as

$$\sum_{i=1}^{n} \psi_x(a_i^n) - \sum_{i=1}^{n-k} \psi_x(c_i^{n-k}) = J_k(x). \tag{5}$$

We now examine the eigenvalues $c_i^{n-k}$. In view of the form of $C_{n-k}$ (made up of diagonal components $B_q$) the numbers $c_i^{n-k}$ are the eigenvalues of $B_q$ for various $q$. Indeed, if $m_{n,q}$ is the number of times the block $B_q$ appears in $C_{n-k}$ then

$$\sum_{i=1}^{n-k} \psi_x(c_i^{n-k}) = \sum_{q=1}^{n-k} m_{n,q} \sum_{i=1}^{q} \psi_x(b_i^a). \tag{6}$$

We also observe the results

$$\sum_{q=1}^{n-k} q m_{n,q} = n-k \tag{7}$$

and

$$k \geqslant \left( \sum_{q=1}^{n-k} m_{n,q} \right) - 1 \geqslant 0$$

and from the latter it follows that

$$\frac{1}{n-k} \sum_{q=1}^{n-k} m_{n,q} \to 0 \qquad \text{as} \quad n \to \infty. \tag{8}$$

Now from (5) and (6) we get

$$\frac{1}{n} \sum_{i=1}^{n} \psi_x(a_i^n) - \frac{n-k}{n} \left\{ \frac{1}{n-k} \sum_{q=1}^{n-k} q m_{n,q} L_q(\psi_x) \right\} = \frac{J_k(x)}{n}, \tag{9}$$

where

$$L_q(\psi_x) = \frac{1}{q} \sum_{i=1}^{q} \psi_x(b_i^q).$$

By (4), $J_k = o(n)$ so it will suffice to show that the expression in brackets in (9) tends to $L(\psi_x)$ (see (3)) as $n \to \infty$. That is, using (7), we have to show that

$$\frac{1}{n-k} \sum_{q=1}^{n-k} q m_{n,q} [L_q(\psi_x) - L(\psi_x)] \to 0.$$

But this is a simple consequence of (7), (8), and the facts that

$$0 \leqslant L_q(\psi_x) \leqslant 1, \qquad 0 \leqslant L(\psi_x) \leqslant 1$$
$$L_q(\psi_x) \to L(\psi_x) \qquad \text{as} \quad q \to \infty.$$

This completes the proof of the theorem.

*Note.* In the theorem above the hypothesis that the limits $\lim_{x \to \pm\infty} f(x)$ exist could be replaced by requiring the two sequences $\{\alpha_n\}$ and $\{\beta_n\}$ to be bounded. For in that case, all of the $x_i^n$ will lie in some compact interval. The proof will be unchanged.


## Acknowledgment

In conclusion, I thank one of the editors for pointing out an omission in the statement of the above theorem.


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